AUTOMATED THEOREM PROVING

Final Exam

Exercise 1. Let $\Phi = \{\neg P(x) \vee Q(f(x), x), P(g(c)), \neg Q(y, z)\}$. Prove that α_Φ is unsatisfiable by finding an unsatisfiable finite set of ground instances of Φ.

Solution: Let $\sigma = \{g(c)/x, f(g(c))/y, g(c)/z\}$. Clearly, $\Phi\sigma = \{\neg P(g(c)) \vee Q(f(g(c)), g(c)), P(g(c)), \neg Q(f(g(c)), g(c))\}$ is unsatisfiable. So, by Herbrand's theorem, α_Φ is unsatisfiable.

Exercise 2. Find all resolvents of the following two clauses:

$\varphi_1 = \neg P(x, y) \vee \neg P(f(x), y) \vee \neg P(f(a), g(u, b)) \vee Q(x, u)$,

$\varphi_2 = P(f(x), g(a, b)) \vee \neg Q(f(a), b)$.

Solution: First, we replace the variable x in φ_2 with a new variable w. We distinguish the following cases.

Case 1. $L = \{\neg P(x, y)\}$, $M = \{P(f(x), g(a, b))\}$ and $N = \{P(x, y), P(f(w), g(a, b))\}$.

By using the unification algorithm, we see that N is unifiable by $\sigma_N = \{f(w)/x, g(a, b)/y\}$. Hence, we obtain the resolvent

$$\neg P(f(f(w)), g(a, b)) \vee \neg P(f(a), g(u, b)) \vee Q(f(w), u) \vee \neg Q(f(a), b).$$

Case 2. $L = \{\neg P(f(x), y)\}$, $M = \{P(f(x), g(a, b))\}$ and $N = \{P(f(x), y), P(f(w), g(a, b))\}$.

By using the unification algorithm, we see that N is unifiable by $\sigma_N = \{w/x, g(a, b)/y\}$. Hence, we obtain the resolvent

$$\neg P(w, g(a, b)) \vee \neg P(f(a), g(u, b)) \vee Q(w, u) \vee \neg Q(f(a), b).$$

Case 3. $N = \{P(f(a), g(u,b)), P(f(w), g(a,b))\}$.

We see that N is unifiable by $\sigma_N = \{a/w, a/u\}$. Hence, we obtain the resolvent

$$\neg P(x,y) \vee \neg P(f(x),y) \vee Q(x,a) \vee \neg Q(f(a),b).$$

Case 4. $N = \{P(x,y), P(f(x),y), P(f(w), g(a,b))\}$.

N is not unifiable, because x and $f(x)$ can't match.

Case 5. $N = \{P(x,y), P(f(a), g(u,b)), P(f(w), g(a,b))\}$.

We see that N is unifiable by $\sigma_N = \{f(a)/x, a/w, a/u, g(a,b)/y\}$. Hence, we obtain the resolvent

$$\neg P(f(f(a)), g(a,b)) \vee Q(f(a),a) \vee \neg Q(f(a),b).$$

Case 6. $N = \{P(f(x),y), P(f(a), g(u,b)), P(f(w), g(a,b))\}$.

N is unifiable by $\sigma_N = \{a/x, a/w, a/u, g(a,b)/y\}$. Hence, we obtain the resolvent

$$\neg P(a, g(a,b)) \vee Q(a,a) \vee \neg Q(f(a),b).$$

Case 7. $N = \{P(x,y), P(f(x),y), P(f(a), g(u,b)), P(f(w), g(a,b))\}$.

N is not unifiable by Case 4.

Case 8. $N = \{Q(x,u), Q(f(a),b)\}$.

N is unifiable by $\sigma_N = \{f(a)/x, b/u\}$. Hence, we obtain the resolvent

$$\neg P(f(a),y) \vee \neg P(f(f(a)),y) \vee \neg P(f(a), g(b,b)) \vee P(f(w), g(a,b)).$$

Exercise 3. Prove by resolution that the formula φ is a logic consequence of the set of formulas $\{\varphi_1, \varphi_2\}$ where:

$\varphi_1 = \exists x(P(x) \wedge \forall y(D(y) \to Q(x,y)))$,

$\varphi_2 = \forall x(P(x) \to \forall y(C(y) \to \neg Q(x,y))$,

$\varphi = \forall x(D(x) \to \neg C(x))$.

Solution: We have to prove by resolution that the set $\{\varphi_1, \varphi_2, \neg\varphi\}$ is unsatisfiable. For this, we have to find standard Skolem forms for φ_1, φ_2 and $\neg\varphi$. Clearly, $\varphi_1 \equiv \exists x \forall y (P(x) \wedge (\neg D(y) \vee Q(x, y)))$. So, the formula $\forall y (P(a) \wedge (\neg D(y) \vee Q(a, y)))$ is a standard Skolem form of φ_1. Also, we have $\varphi_2 \equiv \forall x \forall y (\neg P(x) \vee \neg C(y) \vee \neg Q(x, y))$, which is in standard Skolem form. And $\neg\varphi \equiv \exists x (D(x) \wedge C(x))$, and hence the formula $D(b) \wedge C(b)$ is a standard Skolem form of $\neg\varphi$. Now, from the clauses of the above Skolem forms we give the following proof of $\Box$ by resolution:

1)	$P(a)$	input
2)	$\neg D(y) \vee Q(a, y)$	input
3)	$\neg P(x) \vee \neg C(y) \vee \neg Q(x, y)$	input
4)	$D(b)$	input
5)	$C(b)$	input
6)	$Q(a, b)$	(2,4)
7)	$\neg C(y) \vee \neg Q(a, y)$	(1,3)
8)	$\neg Q(a, b)$	(5,7)
9)	$\Box$	(6,8)

Exercise 4. Write a Prolog program for the predicate delete$(X, L1, L2) \leftarrow$ "$L2$ is the list obtained by deleting in the list $L1$ every occurrence of X".

Solution:

delete$(X, [\,], [\,])$.

delete$(X, [X|L1], L2) :- !, $ delete$(X, L1, L2)$.

delete$(X, [Y|L1], [Y|L2]) :-$ delete$(X, L1, L2)$.

Exercise 5. Ackermann's function is defined for every pair of natural numbers by means of the following equations:

$a(0, y) = y + 1,$
$a(x, 0) = a(x - 1, 1)$ for $x > 0,$
$a(x, y) = a(x - 1, a(x, y - 1))$ for $x, y > 0.$

It is known that Ackermann's function is an example of a recursive function that is not primitive recursive. Then, write a Prolog program to compute Ackermann's function.

Solution:

$\text{ackermann}(0, Y, Z) :- Z \text{ is } Y + 1.$

$\text{ackermann}(X, 0, Z) :- X > 0,\ X1 \text{ is } X - 1,\ \text{ackermann}(X1, 1, Z).$

$\text{ackermann}(X, Y, Z) :- X > 0,\ Y > 0,\ X1 \text{ is } X - 1,\ Y1 \text{ is } Y - 1,$
$\text{ackermann}(X, Y1, Z1), \text{ackermann}(X1, Z1, Z).$